A VARIANCE-REDUCTION TECHNIQUE FOR PRICING DERIVATIVES

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Abstract

Since the introduction of Antithetic Variates by Hammersley and Mortonal in 1956, variance-reduction techniques have flourished in the literature. These techniques improve the efficiency of Monte Carlo (MC) methods by reducing the variance of the distribution of possible outcomes of a simulated random event. Applications of MC methods are founds in physics, mathematics and finance, especially when closed-form solutions are not available. In this paper, we present an original variance-reduction technique that speeds up MC simulations applied to the pricing of exotic options. We test the accuracy of the model with an analytical solution and a naïve numerical model. We show that the model accurately prices a type of exotic options. A wide range of variance reduction techniques is available to speed up MC methods but each technique has its pros and cons when practitioners look at choosing the best method to solve a particular problem. We show that our technique prices with accuracy a type of exotic options and reduces drastically the execution time of the program compared to naïve MC models. Moreover, its implementation is easy and can be applicable in physics, mathematics or finance in various problems such as estimation, optimization, numerical integration or generation of draws from a probability distribution.

Keywords: Monte Carlo Simulation; Option Pricing; Discrete Double Barrier Options; Variance Reduction Technique.

JEL classification: C52; C63; G13.

1. Introduction

Since the introduction of Antithetic Variates by Hammersley and Mortonal in 1956, variance-reduction techniques have flourished in the literature. These techniques improve the efficiency of Monte Carlo (MC) methods by reducing the variance of the distribution of possible outcomes of a simulated random event. Applications of MC methods are founds in physics, mathematics and finance, especially when closed-form solutions are not available. We introduce a variance-reduction technique that speeds up MC simulations when pricing a type of exotic options. Section 2 will review the literature concerning variance reduction and equivalent techniques. Section 3 will present the methodology in five steps. Section 4 will present the results and section 5 will wrap up our findings.

2. Literature review

Many authors have promoted and enhanced Variance-Reduction Techniques (VRTs) to speed up MC simulations significantly. Although computer speed has been increasing dramatically, VRTs are a must in MC simulation (Kleijnen et al., 2010). Faster computers have encouraged practitioners to develop more realistic simulation models; the consequence is that simulation experiments are not as fast as expected. In addition, some simulation models try to capture rare events with very small probabilities of occurrence, so even fast computers would take years to execute a single run. Fortunately, VRTs can reduce significantly execution time of programs that simulate these rare events. We list in Table 1 major works on VRTs:

Table 1: Leading variance reduction techniques

Variance Reduction Techniques	**Applications in Finance**
Antithetic variates (Hammersley and Mortona1, 1956)	Pricing of plain-vanilla and exotic options (Clewlow and Carverhill, 1992)
Control variates (Boyle, 1977)	Pricing of plain-vanilla and exotic options (Hull and White, 1988; Clewlow and Carverhill, 1994)
Stratified sampling (Lepage, 1978)	Value-at-risk and pricing of path-dependent options (Curran, 1994, Glasserman, 2003); pricing Asian options (Fusai and Roncoroni, 2008)
Latin hypercube sampling (McKaya et al., 1979)	Value-at-risk (Härdle et al., 2002); pricing Asian options (Fusai and Roncoroni, 2008)
Common random numbers (Wright and Ramsay, 1979)	
Measure Transformation method (Milstein, 1988)	
Conditioning (McLeish and Rollans, 1992)	
Sequential Monte Carlo Methods (Liu and Chen, 1998)	
Moment matching methods (Kloeden and Platen, 1999)	Value-at-risk (Härdle et al., 2002); pricing Asian options (Fusai and Roncoroni, 2008)
Splitting techniques (Glasserman et al., 1999; Asmussen and Glynn, 2007)	
Least-Squares Monte Carlo method	Pricing American options (Longstaff and Schwartz, 2001)
Quasi-Monte Carlo sampling (Glasserman, 2003)	
Importance sampling (Dupuis et al., 2007), Glasserman et al. (2000)	Value-at-risk (Härdle et al., 2002); pricing barrier options (Salta, 2008)
Multilevel Monte Carlo Path Simulation (Giles, 2008)	

In probabilistic terms, the convergence of the MC method is tracked by the standard deviation of the error which decreases as a square root in terms of the required number of simulations (Korn et al., 2010). The method which reduces at a faster pace the variance of simulation estimates will achieve a most wanted accuracy with less simulation runs: it is called a variance reduction method. Upton and Cook (2008) define VRTs as 'methods for reducing the size of a simulation by efficient use of pseudo-random numbers to reduce the variance of the simulation estimates.' Thus, we can classify the model that we present in this paper as a VRT since it reduces the variance of the simulation estimates.

Beside applications in finance presented in Table 1, VRTs are extensively used for industrial applications. For example, a paper of Perninge et al. (2008) compare four of the most commonly used variance reduction techniques to estimate the trade-off between trading and security in a three-area electric power system: Dagger Sampling, Importance Sampling, Stratified Sampling and the Control Variates method. In Dagger sampling a single random number is used to generate a large number of outcomes. The authors show that Dagger Sampling gives a result that seems very good but actually miss some very important scenarios, yielding a cardinal error. Their paper shows that Stratified Sampling and Control Variates method gave the best results. Other authors (Sabuncuoglu et al., 2008) consider three different types of systems (M/M/1, serial production line and (s, S) inventory control systems) and compare the VRTs under various experimental conditions. They observe that a variance reduction cannot be guaranteed for every instance a VRT is applied. They show that Control Variates and Poststratified Sampling are better on average than Antithetic Variates and Latin Hypercube Sampling. More interestingly, the less-sophisticated techniques (Antithetic Variates and Control Variates) often perform better than the relatively more-complex techniques (Latin Hypercube Sampling and Poststratified Sampling).

Speeding up MC simulations is not only a matter of VRTs: Kleijnen et al. (2010) affirm that '1) Simulation procedures cannot only be speeded up by reducing the variance of the estimator. Careful implementation and storage management should also be optimized to save computing time. 2) Implementing and adapting some of the VRTs requires quite some effort in programming and mathematical considerations.

The gain in variance reduction should also be judged against this additional effort. To put it clearly, is it really worth using a variance reduction method in a specific situation?' Extrapolating Kleijnen et al., there may be other ways of reducing the speed of MC simulations. For example Tian and Benkrid (2010) emulate the hardware itself by applying an FPGA hardware architecture to the acceleration of the Least Squares MC method, with Quasi-Monte Carlo path generation, when pricing American options. A field-programmable gate array (FPGA) is an integrated circuit designed to be configured by a customer or a designer after manufacturing – hence "field-programmable". Tian and Benkrid show that their innovation provides overall speed-up of 20× compared to a CPU-based implementation.

By reviewing VRTs, Platen and Bruti-Liberati (2010) reported two recurrent themes: '1) The greatest gains in efficiency from VRTs result from exploiting specific features of a problem, rather than from generic applications of generic methods. 2) Reducing simulation error is often at odds with convenient estimation of the simulation error itself; in order to supplement a reduced-variance estimator with a valid confidence interval, practitioners sometimes need to sacrifice some of the potential variance reduction.' To interpret Platen and Bruti-Liberati, we cannot generalize the applications of VRTs but choose the most adequate VRT to a particular problem. Besides, estimating the simulation error is subject to the nature of the VRT. Platen and Bruti-Liberati concluded that variance reduction is more of an art and can be applied in many ways.

3. Methodology

We price 3-month and 6-month European double barrier equity call and put options monitored daily with a Logarithmic Increment (Login) model. The Login model speeds up the MC simulation. The Login algorithm repeats each experiment with an increasing number of trials at a logarithmic rate and calculates a weighted average of the results (i.e. options values). We assume that the option value computed at each experiment carries some information on the exact value of the option. The greater the number of trials (i.e. the number of trajectories of the underlying simulated asset), the more accurate is the information about the final option price based on the law of large numbers (LLN, Tchebichef 1846). According to the law, the average of the results obtained from a large number of trials should be close to the expected value, and will tend to become closer as more trials are performed. Our research assumption is the following: the way we increase the number of trials from one experiment to the next one has a deterministic effect on the quality of the information obtained at each simulation. For example, we may choose to increment the number of trials at a logarithmic rate, rather than exponential or linear rate. Furthermore, we assume that the compound information obtained from averaging the results (weighted average versus equally-weighted average) obtained from several experiments offers more information about the exact option price than the information carried away by a single experiment repeated a great number of times. We test the consistency of the Login model with an analytical solution (Ikeda and Kunitomo 1992) adjusted for discretization by Broadie et al. (1997) and the naïve MC simulation presented by Clewlow and Strickland (2000) adapted to the pricing of double barrier options.

We use the parity conditions to compute the Double Barrier Up-and-In-Down-and-In call and put options with the analytical solution of Ikeda and Kunitomo (1992) and the MC simulation models:

- Double Barrier Up-and-In-Down-and-In Call premium = Plain-vanilla Call option premium - Double Barrier Up-and-Out-Down-and-Out Call premium.
- Double Barrier Up-and-In-Down-and-In Put premium = Plain-vanilla Put option premium - Double Barrier Up-and-Out-Down-and-Out Put premium.

We detail the methodology in five steps.

3.1 Step 1

We use the basic algorithm of a naïve MC simulation borrowed from Clewlow and Strickland (2000). The Login model is grafted on the naïve MC simulation which needs minor change to be improved as explained below.
We simulate the trajectory of the stock price using the following Brownian motion:

$$S_{t+1} = S_t . e^{\left(\left(r-\delta-0.5\sigma^2\right)dt+\sigma\sqrt{dt}.\varepsilon\right)} \qquad (1)$$

We place upper and lower barriers that deactivate the option if a barrier is hit (Up-and-Out-Down-and-Out) or activate the option if a barrier is hit (Up-and-In-Down-and-In).

We price European Double Barrier Up-and-Out-Down-and-Out put and call options monitored daily by discounting the payoff of the option at maturity with a discount rate r = 10%:

(2) $$C_0 = e^{-rT} E^Q \left[(S(T) - X)^+ \right]$$

However, basic MC estimates of the expectation of a payoff function may be costly in terms of time and computer resource usage: this fact will be illustrated in the results section by the high execution time of naïve MC methods. We introduce three incremental methods besides naïve MC methods where the MC simulation experiment is repeated ten times by increasing the number of trajectories N of the simulated stock price following three different processes as described at step 2. The maturity of the option that we price is either T= 0.25 year (63 steps) or T= 0.5 year (125 steps), assuming a year equal to 250 business days. The other parameters of the option are the underlying stock price S = 100, the strike price X = 100, the 3 month or 6-month interest rate r = 0.10, the stock dividend q = 0, and the rebate of the option R = 0.

3.2 Step 2

The total number of experiments is ten. For each new experiment, we increment the number of trials (i.e. trajectories of the underlying stock) using three processes, either following:

- a linear increment (*Linearin* Model) with
$y = 1255.236x$ *(x = 1 to 10)* (3)
- an exponential increment (*Expin* model) with
$y = 1091.636e^{(0.28x)}$ *(x = 1 to 10)* (4)
- a logarithmic increment (*Login* model) with
$y=3908.7*Log_e(x)+1000$ *(x = 1 to 10)* (5)

Figure 1 and Table 2 illustrate the three processes.
Figure 1: Linear, Exponential and Logarithmic Increments of trials (i.e. simulated trajectories)

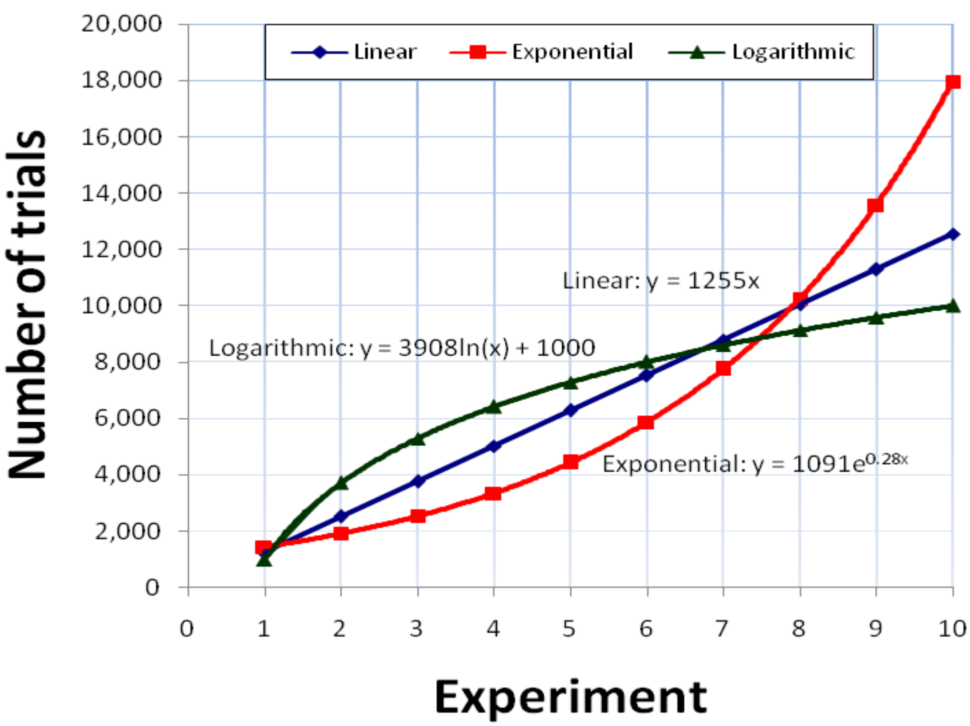

Table 2: Linear, Exponential and Logarithmic Increments
of trials (i.e. simulated trajectories)

		Number of trials	
Experiment	Linear	Exponential	Logarithmic
1	1255	1444	1000
2	2510	1911	3709
3	3766	2529	5294
4	5021	3346	6419
5	6276	4427	7291
6	7531	5857	8003
7	8787	7750	8606
8	10042	10254	9128
9	11297	13568	9588
10	12552	17952	10000
Total trials	69038	69038	69038

3.3 Step 3

Once we have computed 10 options prices b_i at experiments 1 to 10, at the discretion of the practitioner when writing the algorithm, he (she) may apply one of the three following methods that converge toward the same option price x:

1) x? | Min $\left[y = \frac{1}{\sum a_i} \sum_{i=1}^{10} a_i (b_i - x)^2\right]$ (6)

2) x? | $\left[y = \frac{1}{\sum a_i} \sum_{i=1}^{10} a_i (b_i - x) = 0\right]$ (7)

3) $x = \frac{1}{\sum a_i} \sum_{i=1}^{10} (a_i b_i)$ (8)

with a_i = number of trials at experiment i
and b_i = option price obtained at experiment i

3.4 Step 4

We compare the option price obtained with the Login model to the ones computed with the naïve MC simulation in two ways: 1) with one experiment only and N = 69,038 trials (i.e. trajectories) or 2) with 10 experiments (i.e. batches) of N = 6,904 trials.

3.5 Step 5

The benchmark is the analytical solution (Ikeda and Kunitomo 1992) adjusted for discretization by Broadie et al. (1997).

The model involves two parameters δ_1 and δ_2 which determine the curvature of the lower L and upper U absorbing boundaries. Our paper covers three possible cases to compute double barrier options:
1) $\delta_1 = \delta_2 = 0$, which corresponds to two flat boundaries.

2) $\delta_1 < 0 < \delta_2$, which corresponds to a lower boundary exponentially growing as time elapses, while the upper boundary will be exponentially decaying.

3) $\delta_1 > 0 > \delta_2$, which corresponds to a convex downward lower boundary and a convex upward upper boundary.
Cases 2) and 3) are respectively lower and higher bounds of case 1).

The Linearin, Expin and Login model as well as the other MC models assume that the crossing of double barriers is monitored daily by dividing the maturity of the option of 3(6) months in 63(125) steps assuming 1 year = 250 steps.

The analytical solution of Ikeda and Kunitomo (1992) assumes that the crossing of barriers is monitored continuously. As mentioned in the literature review, Broadie et al. (1997) developed an approximation for continuity correction to price discrete barrier options. The correction shifts the barrier away from the underlying asset price that reduces the probability of hitting the barrier and that mimics the effect of discrete monitoring that lowers the probability of hitting the barrier compared to continuous monitoring. The barrier H_{AC}, After Correction is equal to:

$$H_{AC} = H.e^{\beta\sigma\sqrt{\Delta t}} \qquad (9)$$

When $H>S_0$

And is equal to: $H_{AC} = H.e^{-\beta\sigma\sqrt{\Delta t}}$ (10) When $H<S_0$

With σ = volatility (either 0.15, 0.25 or 0.35 in our example), H the initial barrier, $\beta = -\frac{\zeta\left(\frac{1}{2}\right)}{\sqrt{2\pi}} = 0.5826$, where ζ is the Riemann zeta function. For example, Δt = 0.5/125 = 0.004 when the option maturity is 6 months divided in 125 days -steps).

All MC models are benchmarked to the Ikeda and Kunitomo (1992) model adjusted for discrete time by Broadie et al. (1997) using the Root Mean Square Error criteria (RMSE).

4. Results

Tables 3, 4, 5 and 6 gather the computed premiums of European double barrier call and put options, respectively for Double Barrier Up-and-Out-Down-and-Out Call Options, Double Barrier Up-and-Out-Down-and-Out Put Options, Double Barrier Up-and-In-Down-and-In Call Options, Double Barrier Up-and-In-Down-and-In Put Options. In addition, the tables display the average execution time (simulation time), i.e. the average time the program is running. Our computer's central processing unit (CPU) is the Intel Core 2 Duo E8400 @3.00GHz 3.00GHz, with a RAM of 1GB. We price a double barrier option where S = 100, X = 100, r =0.10, q = dividend = 0, rebate R = 0, σ = 0.15, 0.25 or 0.35 and T = 0.25 or 0.5. We compute the option premiums for different levels of barrier down L and barrier up U as suggested in Haug (2006).

Table 3: Premiums of European Double Barrier Up-and-Out-Down-and-Out Call Options (S =100, X=100, r=0.10, q=0, rebate = 0) computed with analytical and numerical models

				Ikeda and Kunimoto Continuous recording						Ikeda and Kunimoto Daily recording (Broadie et al.)						Expin model-Daily recording (25-second average simulation time)						Login model-Daily recording (25-second average simulation time)						Linearin model-Daily recording (25-second average simulation time)						Naïve MC-Daily recording: 10 batches of 6,904 simulations (220-second average simulation time)						Naïve MC-Daily recording 69,038 simulations (220-second average simulation time)					
				T=0.25			T=0.50			T=0.25			T=0.50			T=0.25			T=0.50			T=0.25			T=0.50			T=0.25			T=0.50			T=0.25			T=0.50			T=0.25			T=0.50		
				σ			σ			σ			σ			σ			σ			σ			σ			σ			σ			σ			σ			σ			σ		
L	U	δ1	δ2	0.15	0.25	0.35	0.15	0.25	0.35	0.15	0.25	0.35	0.15	0.25	0.35	0.15	0.25	0.35	0.15	0.25	0.35	0.15	0.25	0.35	0.15	0.25	0.35	0.15	0.25	0.35	0.15	0.25	0.35	0.15	0.25	0.35	0.15	0.25	0.35	0.15	0.25	0.35	0.15	0.25	0.35
50	150	0	0	4.35	6.16	7.04	6.99	7.93	6.51	4.35	6.18	7.20	6.99	8.09	6.89	4.35	6.32	7.14	6.92	8.07	6.72	4.42	6.22	7.24	7.02	8.03	6.83	4.36	6.38	7.20	6.99	8.00	6.98	4.43	6.24	7.15	7.08	8.10	6.84	4.38	6.20	7.26	7.03	8.12	6.83
60	140	0	0	4.35	5.85	5.77	6.81	6.34	4.38	4.35	5.92	6.06	6.84	6.59	4.78	4.42	5.93	6.12	6.88	6.75	4.63	4.43	5.93	6.06	6.83	6.58	4.72	4.42	5.76	5.97	6.81	6.54	4.79	4.38	5.92	6.07	6.85	6.57	4.76	4.39	5.95	6.03	6.87	6.61	4.74
70	130	0	0	4.31	4.83	3.78	5.97	4.00	2.26	4.32	5.01	4.16	6.07	4.31	2.59	4.41	5.05	4.07	6.09	4.18	2.57	4.34	5.05	4.09	6.12	4.31	2.54	4.37	4.90	4.20	6.06	4.29	2.67	4.33	4.96	4.10	6.08	4.24	2.52	4.33	5.00	4.09	6.07	4.25	2.55
80	120	0	0	3.75	2.64	1.49	3.58	1.51	0.56	3.84	2.93	1.82	3.79	1.76	0.76	3.91	2.91	1.81	3.85	1.71	0.72	3.85	2.90	1.77	3.76	1.75	0.77	3.88	2.91	1.77	3.92	1.78	0.75	3.84	2.88	1.81	3.77	1.73	0.73	3.86	2.86	1.77	3.74	1.71	0.74
90	110	0	0	1.21	0.31	0.05	0.55	0.04	0.00	1.39	0.47	0.12	0.69	0.09	0.01	1.36	0.46	0.11	0.67	0.09	0.01	1.38	0.45	0.11	0.68	0.08	0.01	1.40	0.44	0.12	0.66	0.09	0.01	1.37	0.45	0.11	0.68	0.09	0.01	1.36	0.45	0.11	0.67	0.09	0.01
50	150	-0.1	0.1	4.35	6.10	6.70	6.90	6.98	5.21	4.35	6.13	6.90	6.91	7.20	5.61	4.35	6.32	7.14	6.92	8.07	6.72	4.42	6.22	7.24	7.02	8.03	6.83	4.36	6.38	7.20	6.99	8.00	6.98	4.43	6.24	7.15	7.08	8.10	6.84	4.38	6.20	7.26	7.03	8.12	6.83
60	140	-0.1	0.1	4.35	5.64	5.25	6.41	5.02	3.15	4.35	5.73	5.56	6.48	5.31	3.52	4.42	5.93	6.12	6.88	6.75	4.63	4.43	5.93	6.06	6.83	6.58	4.72	4.42	5.76	5.97	6.81	6.54	4.79	4.38	5.92	6.07	6.85	6.57	4.76	4.39	5.95	6.03	6.87	6.61	4.74
70	130	-0.1	0.1	4.26	4.33	3.15	4.82	2.63	1.34	4.27	4.55	3.53	4.99	2.91	1.61	4.41	5.05	4.07	6.09	4.18	2.57	4.34	5.05	4.09	6.12	4.31	2.54	4.37	4.90	4.20	6.06	4.29	2.67	4.33	4.96	4.10	6.08	4.24	2.52	4.33	5.00	4.09	6.07	4.25	2.55
80	120	-0.1	0.1	3.30	1.99	1.04	1.92	0.65	0.18	3.42	2.27	1.32	2.12	0.81	0.29	3.91	2.91	1.81	3.85	1.71	0.72	3.85	2.90	1.77	3.76	1.75	0.77	3.88	2.91	1.77	3.92	1.78	0.75	3.84	2.88	1.81	3.77	1.73	0.73	3.86	2.86	1.77	3.74	1.71	0.74
90	110	-0.1	0.1	0.59	0.10	0.01	0.04	0.00	0.00	0.73	0.19	0.04	0.07	0.00	0.00	1.36	0.46	0.11	0.67	0.09	0.01	1.38	0.45	0.11	0.68	0.08	0.01	1.40	0.44	0.12	0.66	0.09	0.01	1.37	0.45	0.11	0.68	0.09	0.01	1.36	0.45	0.11	0.67	0.09	0.01
50	150	0.1	-0.1	4.35	6.20	7.32	7.01	8.61	7.72	4.35	6.21	7.45	7.01	8.72	8.07	4.35	6.32	7.14	6.92	8.07	6.72	4.42	6.22	7.24	7.02	8.03	6.83	4.36	6.38	7.20	6.99	8.00	6.98	4.43	6.24	7.15	7.08	8.10	6.84	4.38	6.20	7.26	7.03	8.12	6.83
60	140	0.1	-0.1	4.35	6.00	6.24	6.96	7.43	5.66	4.35	6.05	6.49	6.97	7.63	6.07	4.42	5.93	6.12	6.88	6.75	4.63	4.43	5.93	6.06	6.83	6.58	4.72	4.42	5.76	5.97	6.81	6.54	4.79	4.38	5.92	6.07	6.85	6.57	4.76	4.39	5.95	6.03	6.87	6.61	4.74
70	130	0.1	-0.1	4.34	5.24	4.39	6.61	5.38	3.34	4.34	5.38	4.75	6.66	5.67	3.73	4.41	5.05	4.07	6.09	4.18	2.57	4.34	5.05	4.09	6.12	4.31	2.54	4.37	4.90	4.20	6.06	4.29	2.67	4.33	4.96	4.10	6.08	4.24	2.52	4.33	5.00	4.09	6.07	4.25	2.55
80	120	0.1	-0.1	4.04	3.29	2.00	5.07	2.66	1.19	4.09	3.57	2.37	5.24	2.97	1.48	3.91	2.91	1.81	3.85	1.71	0.72	3.85	2.90	1.77	3.76	1.75	0.77	3.88	2.91	1.77	3.92	1.78	0.75	3.84	2.88	1.81	3.77	1.73	0.73	3.86	2.86	1.77	3.74	1.71	0.74
90	110	0.1	-0.1	1.92	0.65	0.14	1.71	0.30	0.03	2.13	0.87	0.29	1.93	0.46	0.07	1.36	0.46	0.11	0.67	0.09	0.01	1.38	0.45	0.11	0.68	0.08	0.01	1.40	0.44	0.12	0.66	0.09	0.01	1.37	0.45	0.11	0.68	0.09	0.01	1.36	0.45	0.11	0.67	0.09	0.01
															RMSE:	0.29	0.32	0.36	0.76	0.83	0.79	0.29	0.32	0.36	0.75	0.82	0.79	0.29	0.34	0.36	0.76	0.82	0.79	0.29	0.32	0.36	0.75	0.83	0.79	0.29	0.32	0.36	0.75	0.83	0.79
																		RMSE:	0.6080						0.6032						0.6053						0.6033						0.6036		

Table 4: Premiums of European Double Barrier Up-and-Out-Down-and-Out Put Options (S =100, X=100, r=0.10, q=0, rebate = 0) computed with analytical and numerical models

				Ikeda and Kunimoto Continuous recording						Ikeda and Kunimoto Daily recording (Broadie et al.)					
				T=0.25			T=0.50			T=0.25			T=0.50		
				σ			σ			σ			σ		
L	U	δ1	δ2	0.15	0.25	0.35	0.15	0.25	0.35	0.15	0.25	0.35	0.15	0.25	0.35
50	150	0	0	1.88	3.79	5.72	2.14	4.70	7.17	1.88	3.79	5.72	2.14	4.70	7.20
60	140	0	0	1.88	3.78	5.61	2.14	4.62	6.11	1.88	3.78	5.63	2.14	4.64	6.24
70	130	0	0	1.88	3.70	4.65	2.13	3.89	3.59	1.88	3.72	4.80	2.13	3.98	3.87
80	120	0	0	1.86	2.69	2.07	1.89	1.79	0.82	1.86	2.83	2.39	1.92	1.98	1.06
90	110	0	0	0.95	0.34	0.06	0.46	0.05	0.00	1.04	0.50	0.14	0.54	0.10	0.01
50	150	-0.1	0.1	1.88	3.79	5.72	2.14	4.70	7.03	1.88	3.79	5.72	2.14	4.70	7.08
60	140	-0.1	0.1	1.88	3.78	5.55	2.14	4.53	5.57	1.88	3.78	5.58	2.14	4.55	5.75
70	130	-0.1	0.1	1.88	3.64	4.35	2.11	3.38	2.70	1.88	3.67	4.53	2.12	3.51	3.00
80	120	-0.1	0.1	1.83	2.32	1.61	1.54	1.01	0.33	1.84	2.49	1.92	1.59	1.19	0.49
90	110	-0.1	0.1	0.58	0.13	0.01	0.05	0.00	0.00	0.68	0.23	0.05	0.09	0.00	0.00
50	150	0.1	-0.1	1.88	3.79	5.72	2.14	4.70	7.25	1.88	3.79	5.72	2.14	4.70	7.27
60	140	0.1	-0.1	1.88	3.79	5.64	2.14	4.67	6.51	1.88	3.79	5.66	2.14	4.68	6.61
70	130	0.1	-0.1	1.88	3.74	4.90	2.14	4.24	4.41	1.88	3.75	5.02	2.14	4.30	4.66
80	120	0.1	-0.1	1.87	2.99	2.54	2.05	2.55	1.49	1.88	3.10	2.85	2.07	2.73	1.79
90	110	0.1	-0.1	1.27	0.65	0.16	1.03	0.29	0.03	1.35	0.86	0.31	1.12	0.42	0.07

				Expin model-Daily recording (25-second average simulation time)						Login model-Daily recording (25-second average simulation time)					
				T=0.25			T=0.50			T=0.25			T=0.50		
				σ			σ			σ			σ		
L	U	δ1	δ2	0.15	0.25	0.35	0.15	0.25	0.35	0.15	0.25	0.35	0.15	0.25	0.35
50	150	0	0	1.88	3.80	5.85	2.16	4.78	7.26	1.89	3.81	5.75	2.15	4.73	7.24
60	140	0	0	1.90	3.79	5.64	2.15	4.63	6.25	1.90	3.80	5.62	2.17	4.64	6.20
70	130	0	0	1.89	3.77	4.79	2.13	3.95	3.82	1.88	3.71	4.77	2.14	3.95	3.84
80	120	0	0	1.83	2.80	2.39	1.89	1.98	1.03	1.87	2.83	2.34	1.88	1.97	1.04
90	110	0	0	1.04	0.49	0.13	0.53	0.09	0.01	1.04	0.48	0.13	0.54	0.09	0.01
50	150	-0.1	0.1	1.88	3.80	5.85	2.16	4.78	7.26	1.89	3.81	5.75	2.15	4.73	7.24
60	140	-0.1	0.1	1.90	3.79	5.64	2.15	4.63	6.25	1.90	3.80	5.62	2.17	4.64	6.20
70	130	-0.1	0.1	1.89	3.77	4.79	2.13	3.95	3.82	1.88	3.71	4.77	2.14	3.95	3.84
80	120	-0.1	0.1	1.83	2.83	2.39	1.89	1.98	1.03	1.87	2.83	2.34	1.88	1.97	1.04
90	110	-0.1	0.1	1.04	0.49	0.13	0.53	0.09	0.01	1.04	0.48	0.13	0.54	0.09	0.01
50	150	0.1	-0.1	1.88	3.80	5.85	2.16	4.78	7.26	1.89	3.81	5.75	2.15	4.73	7.24
60	140	0.1	-0.1	1.90	3.79	5.64	2.15	4.63	6.25	1.90	3.80	5.62	2.17	4.64	6.20
70	130	0.1	-0.1	1.89	3.77	4.79	2.13	3.95	3.82	1.88	3.71	4.77	2.14	3.95	3.84
80	120	0.1	-0.1	1.83	2.83	2.39	1.89	1.98	1.03	1.87	2.83	2.34	1.88	1.97	1.04
90	110	0.1	-0.1	1.04	0.49	0.13	0.53	0.09	0.01	1.04	0.48	0.13	0.54	0.09	0.01
RMSE:				0.12	0.17	0.21	0.21	0.33	0.42	0.12	0.16	0.20	0.21	0.33	0.42
RMSE:									0.2643						0.2624

				Linearin model-Daily recording (25-second average simulation time)						Naïve MC-Daily recording: 10 batches of 6,904 simulations (220-second average simulation time)						Naïve MC-Daily recording 69,038 simulations (220-second average simulation time)					
				T=0.25			T=0.50			T=0.25			T=0.50			T=0.25			T=0.50		
				σ			σ			σ			σ			σ			σ		
L	U	δ1	δ2	0.15	0.25	0.35	0.15	0.25	0.35	0.15	0.25	0.35	0.15	0.25	0.35	0.15	0.25	0.35	0.15	0.25	0.35
50	150	0	0	1.88	3.79	5.77	2.13	4.81	7.21	1.91	3.82	5.86	2.16	4.69	7.22	1.87	3.79	5.73	2.12	4.73	7.16
60	140	0	0	1.91	3.80	5.63	2.17	4.56	6.26	1.87	3.79	5.65	2.13	4.65	6.25	1.90	3.79	5.67	2.15	4.66	6.28
70	130	0	0	1.83	3.76	4.73	2.08	3.92	3.88	1.89	3.73	4.80	2.15	4.01	3.86	1.91	3.71	4.78	2.13	3.98	3.87
80	120	0	0	1.87	2.83	2.29	1.89	1.92	1.03	1.87	2.81	2.33	1.89	1.97	1.04	1.87	2.84	2.32	1.93	1.96	1.05
90	110	0	0	1.02	0.49	0.14	0.51	0.10	0.01	1.02	0.48	0.14	0.53	0.09	0.01	1.03	0.48	0.14	0.54	0.10	0.01
50	150	-0.1	0.1	1.88	3.79	5.77	2.13	4.81	7.21	1.91	3.82	5.86	2.16	4.69	7.22	1.87	3.79	5.73	2.12	4.73	7.16
60	140	-0.1	0.1	1.91	3.80	5.63	2.17	4.56	6.26	1.87	3.79	5.65	2.13	4.65	6.25	1.90	3.79	5.67	2.15	4.66	6.28
70	130	-0.1	0.1	1.83	3.76	4.73	2.08	3.92	3.88	1.89	3.73	4.80	2.15	4.01	3.86	1.91	3.71	4.78	2.13	3.98	3.87
80	120	-0.1	0.1	1.87	2.83	2.29	1.89	1.92	1.03	1.87	2.81	2.33	1.89	1.97	1.04	1.87	2.84	2.32	1.93	1.96	1.05
90	110	-0.1	0.1	1.02	0.49	0.14	0.51	0.10	0.01	1.02	0.48	0.14	0.53	0.09	0.01	1.03	0.48	0.14	0.54	0.10	0.01
50	150	0.1	-0.1	1.88	3.79	5.77	2.13	4.81	7.21	1.91	3.82	5.86	2.16	4.69	7.22	1.87	3.79	5.73	2.12	4.73	7.16
60	140	0.1	-0.1	1.91	3.80	5.63	2.17	4.56	6.26	1.87	3.79	5.65	2.13	4.65	6.25	1.90	3.79	5.67	2.15	4.66	6.28
70	130	0.1	-0.1	1.83	3.76	4.73	2.08	3.92	3.88	1.89	3.73	4.80	2.15	4.01	3.86	1.91	3.71	4.78	2.13	3.98	3.87
80	120	0.1	-0.1	1.87	2.83	2.29	1.89	1.92	1.03	1.87	2.81	2.33	1.89	1.97	1.04	1.87	2.84	2.32	1.93	1.96	1.05
90	110	0.1	-0.1	1.02	0.49	0.14	0.51	0.10	0.01	1.02	0.48	0.14	0.53	0.09	0.01	1.03	0.48	0.14	0.54	0.10	0.01
RMSE:				0.13	0.16	0.21	0.21	0.33	0.42	0.12	0.16	0.21	0.21	0.33	0.42	0.12	0.16	0.20	0.21	0.33	0.42
RMSE:									0.2648						0.2640						0.2631

Table 5: Premiums of European Double Barrier Up-and-In-Down-and-In Call Options (S =100, X=100, r=0.10, q=0, rebate = 0) computed with analytical and numerical models

				Ikeda and Kunimoto Continuous recording						Ikeda and Kunimoto Daily recording (Broadie et al.)					
				T=0.25			T=0.50			T=0.25			T=0.50		
				σ			σ			σ			σ		
L	U	δ1	δ2	0.15	0.25	0.35	0.15	0.25	0.35	0.15	0.25	0.35	0.15	0.25	0.35
50	150	0	0	0.00	0.09	1.15	0.03	1.65	5.73	0.00	0.07	0.99	0.02	1.49	5.35
60	140	0	0	0.00	0.40	2.42	0.21	3.24	7.86	0.00	0.34	2.13	0.18	2.99	7.46
70	130	0	0	0.04	1.43	4.41	1.04	5.58	9.98	0.03	1.24	4.04	0.94	5.27	9.65
80	120	0	0	0.60	3.62	6.70	3.43	8.07	11.68	0.51	3.33	6.37	3.22	7.82	11.48
90	110	0	0	3.15	5.94	8.14	6.46	9.54	12.24	2.96	5.78	8.07	6.33	9.49	12.23
50	150	-0.1	0.1	0.00	0.15	1.49	0.12	2.60	7.03	0.00	0.13	1.29	0.10	2.38	6.64
60	140	-0.1	0.1	0.00	0.62	2.94	0.60	4.56	9.09	0.00	0.52	2.63	0.54	4.27	8.72
70	130	-0.1	0.1	0.10	1.93	5.04	2.20	6.96	10.90	0.08	1.71	4.66	2.03	6.67	10.63
80	120	-0.1	0.1	1.06	4.27	7.16	5.09	8.94	12.06	0.93	3.99	6.87	4.89	8.77	11.95
90	110	-0.1	0.1	3.76	6.15	8.18	6.97	9.58	12.24	3.62	6.06	8.16	6.94	9.58	12.24
50	150	0.1	-0.1	0.00	0.05	0.88	0.01	0.97	4.52	0.00	0.04	0.74	0.00	0.87	4.17
60	140	0.1	-0.1	0.00	0.25	1.95	0.06	2.16	6.58	0.00	0.21	1.70	0.05	1.96	6.18
70	130	0.1	-0.1	0.01	1.02	3.81	0.41	4.21	8.90	0.01	0.87	3.44	0.36	3.91	8.51
80	120	0.1	-0.1	0.31	2.97	6.19	1.94	6.92	11.05	0.26	2.69	5.82	1.78	6.62	10.76
90	110	0.1	-0.1	2.43	5.61	8.05	5.31	9.28	12.22	2.23	5.38	7.90	5.09	9.13	12.17

				Expin model-Daily recording (25-second average simulation time)						Login model-Daily recording (25-second average simulation time)					
				T=0.25			T=0.50			T=0.25			T=0.50		
				σ			σ			σ			σ		
L	U	δ1	δ2	0.15	0.25	0.35	0.15	0.25	0.35	0.15	0.25	0.35	0.15	0.25	0.35
50	150	0	0	0.00	0.00	1.05	0.09	1.51	5.52	0.00	0.03	0.95	0.00	1.55	5.41
60	140	0	0	0.00	0.33	2.07	0.13	2.84	7.61	0.00	0.32	2.13	0.18	3.00	7.52
70	130	0	0	0.00	1.21	4.12	0.92	5.40	9.67	0.01	1.21	4.10	0.90	5.27	9.70
80	120	0	0	0.44	3.35	6.38	3.16	7.87	11.52	0.51	3.35	6.42	3.25	7.83	11.47
90	110	0	0	2.99	5.79	8.08	6.34	9.49	12.23	2.97	5.80	8.08	6.33	9.50	12.23
50	150	-0.1	0.1	0.00	0.00	1.05	0.09	1.51	5.52	0.00	0.03	0.95	0.00	1.55	5.41
60	140	-0.1	0.1	0.00	0.33	2.07	0.13	2.84	7.61	0.00	0.32	2.13	0.18	3.00	7.52
70	130	-0.1	0.1	0.00	1.21	4.12	0.92	5.40	9.67	0.01	1.21	4.10	0.90	5.27	9.70
80	120	-0.1	0.1	0.44	3.35	6.38	3.16	7.87	11.52	0.51	3.35	6.42	3.25	7.83	11.47
90	110	-0.1	0.1	2.99	5.79	8.08	6.34	9.49	12.23	2.97	5.80	8.08	6.33	9.50	12.23
50	150	0.1	-0.1	0.00	0.00	1.05	0.09	1.51	5.52	0.00	0.03	0.95	0.00	1.55	5.41
60	140	0.1	-0.1	0.00	0.33	2.07	0.13	2.84	7.61	0.00	0.32	2.13	0.18	3.00	7.52
70	130	0.1	-0.1	0.00	1.21	4.12	0.92	5.40	9.67	0.01	1.21	4.10	0.90	5.27	9.70
80	120	0.1	-0.1	0.44	3.35	6.38	3.16	7.87	11.52	0.51	3.35	6.42	3.25	7.83	11.47
90	110	0.1	-0.1	2.99	5.79	8.08	6.34	9.49	12.23	2.97	5.80	8.08	6.33	9.50	12.23
RMSE:				0.29	0.32	0.36	0.76	0.83	0.79	0.28	0.32	0.36	0.75	0.82	0.79
RMSE:									0.6072						0.6029

				Linearin model-Daily recording (25-second average simulation time)						Naïve MC-Daily recording: 10 batches of 6,904 simulations (220-second average simulation time)						Naïve MC-Daily recording 69,038 simulations (220-second average simulation time)					
				T=0.25			T=0.50			T=0.25			T=0.50			T=0.25			T=0.50		
				σ			σ			σ			σ			σ			σ		
L	U	δ1	δ2	0.15	0.25	0.35	0.15	0.25	0.35	0.15	0.25	0.35	0.15	0.25	0.35	0.15	0.25	0.35	0.15	0.25	0.35
50	150	0	0	0.00	0.00	0.99	0.03	1.58	5.26	0.00	0.02	1.04	0.00	1.48	5.40	0.00	0.06	0.94	0.00	1.46	5.41
60	140	0	0	0.00	0.49	2.22	0.20	3.04	7.45	0.00	0.34	2.12	0.17	3.01	7.48	0.00	0.30	2.16	0.14	2.98	7.50
70	130	0	0	0.00	1.35	3.99	0.95	5.30	9.57	0.02	1.29	4.09	0.94	5.34	9.72	0.02	1.26	4.10	0.95	5.33	9.69
80	120	0	0	0.47	3.34	6.42	3.10	7.80	11.49	0.52	3.38	6.38	3.25	7.85	11.51	0.49	3.39	6.42	3.28	7.87	11.50
90	110	0	0	2.96	5.82	8.07	6.36	9.49	12.23	2.98	5.80	8.08	6.33	9.49	12.23	2.99	5.81	8.08	6.34	9.50	12.23
50	150	-0.1	0.1	0.00	0.00	0.99	0.03	1.58	5.26	0.00	0.02	1.04	0.00	1.48	5.40	0.00	0.06	0.94	0.00	1.46	5.41
60	140	-0.1	0.1	0.00	0.49	2.22	0.20	3.04	7.45	0.00	0.34	2.12	0.17	3.01	7.48	0.00	0.30	2.16	0.14	2.98	7.50
70	130	-0.1	0.1	0.00	1.35	3.99	0.95	5.30	9.57	0.02	1.29	4.09	0.94	5.34	9.72	0.02	1.26	4.10	0.95	5.33	9.69
80	120	-0.1	0.1	0.47	3.34	6.42	3.10	7.80	11.49	0.52	3.38	6.38	3.25	7.85	11.51	0.49	3.39	6.42	3.28	7.87	11.50
90	110	-0.1	0.1	2.96	5.82	8.07	6.36	9.49	12.23	2.98	5.80	8.08	6.33	9.49	12.23	2.99	5.81	8.08	6.34	9.50	12.23
50	150	0.1	-0.1	0.00	0.00	0.99	0.03	1.58	5.26	0.00	0.02	1.04	0.00	1.48	5.40	0.00	0.06	0.94	0.00	1.46	5.41
60	140	0.1	-0.1	0.00	0.49	2.22	0.20	3.04	7.45	0.00	0.34	2.12	0.17	3.01	7.48	0.00	0.30	2.16	0.14	2.98	7.50
70	130	0.1	-0.1	0.00	1.35	3.99	0.95	5.30	9.57	0.02	1.29	4.09	0.94	5.34	9.72	0.02	1.26	4.10	0.95	5.33	9.69
80	120	0.1	-0.1	0.47	3.34	6.42	3.10	7.80	11.49	0.52	3.38	6.38	3.25	7.85	11.51	0.49	3.39	6.42	3.28	7.87	11.50
90	110	0.1	-0.1	2.96	5.82	8.07	6.36	9.49	12.23	2.98	5.80	8.08	6.33	9.49	12.23	2.99	5.81	8.08	6.34	9.50	12.23
RMSE:				0.29	0.33	0.36	0.76	0.82	0.79	0.28	0.32	0.36	0.75	0.83	0.79	0.29	0.32	0.36	0.75	0.83	0.79
RMSE:									0.6041						0.6029						0.6035

Table 6: Premiums of European Double Barrier Up-and-In-Down-and-In Put Options
(S =100, X=100, r=0.10, q=0, rebate = 0) computed with analytical and numerical models

				Ikeda and Kunimoto Continuous recording						Ikeda and Kunimoto Daily recording (Broadie et al.)					
				T=0.25			T=0.50			T=0.25			T=0.50		
				σ			σ			σ			σ		
L	U	δ1	δ2	0.15	0.25	0.35	0.15	0.25	0.35	0.15	0.25	0.35	0.15	0.25	0.35
50	150	0	0	0.00	0.00	0.00	0.00	0.00	0.20	0.00	0.00	0.00	0.00	0.00	0.17
60	140	0	0	0.00	0.00	0.12	0.00	0.08	1.26	0.00	0.00	0.09	0.00	0.07	1.12
70	130	0	0	0.00	0.08	1.07	0.00	0.81	3.78	0.00	0.07	0.92	0.00	0.72	3.49
80	120	0	0	0.02	1.10	3.65	0.25	2.92	6.54	0.02	0.96	3.34	0.22	2.73	6.30
90	110	0	0	0.94	3.44	5.66	1.68	4.66	7.36	0.84	3.28	5.58	1.60	4.61	7.36
50	150	-0.1	0.1	0.00	0.00	0.01	0.00	0.01	0.33	0.00	0.00	0.00	0.00	0.00	0.28
60	140	-0.1	0.1	0.00	0.00	0.17	0.00	0.18	1.80	0.00	0.00	0.14	0.00	0.15	1.62
70	130	-0.1	0.1	0.00	0.14	1.37	0.02	1.32	4.66	0.00	0.12	1.19	0.02	1.20	4.37
80	120	-0.1	0.1	0.06	1.46	4.12	0.60	3.69	7.03	0.05	1.30	3.81	0.54	3.51	6.87
90	110	-0.1	0.1	1.31	3.66	5.71	2.08	4.70	7.36	1.21	3.56	5.68	2.05	4.70	7.36
50	150	0.1	-0.1	0.00	0.00	0.00	0.00	0.00	0.11	0.00	0.00	0.00	0.00	0.00	0.09
60	140	0.1	-0.1	0.00	0.00	0.08	0.00	0.04	0.85	0.00	0.00	0.06	0.00	0.03	0.75
70	130	0.1	-0.1	0.00	0.05	0.83	0.00	0.46	2.96	0.00	0.04	0.70	0.00	0.41	2.70
80	120	0.1	-0.1	0.01	0.80	3.18	0.08	2.16	5.87	0.01	0.68	2.87	0.07	1.98	5.57
90	110	0.1	-0.1	0.61	3.13	5.56	1.11	4.42	7.34	0.53	2.93	5.41	1.02	4.29	7.29

				Expin model-Daily recording (25-second average simulation time)						Login model-Daily recording (25-second average simulation time)					
				T=0.25			T=0.50			T=0.25			T=0.50		
				σ			σ			σ			σ		
L	U	δ1	δ2	0.15	0.25	0.35	0.15	0.25	0.35	0.15	0.25	0.35	0.15	0.25	0.35
50	150	0	0	0.00	0.00	0.00	0.00	0.00	0.11	0.00	0.00	0.00	0.00	0.00	0.12
60	140	0	0	0.00	0.00	0.08	0.00	0.07	1.11	0.00	0.00	0.10	0.00	0.07	1.16
70	130	0	0	0.00	0.01	0.93	0.01	0.75	3.54	0.00	0.08	0.95	0.00	0.76	3.53
80	120	0	0	0.05	0.95	3.33	0.25	2.72	6.34	0.01	0.96	3.39	0.25	2.73	6.33
90	110	0	0	0.85	3.30	5.59	1.61	4.62	7.36	0.85	3.30	5.59	1.60	4.62	7.36
50	150	-0.1	0.1	0.00	0.00	0.00	0.00	0.00	0.11	0.00	0.00	0.00	0.00	0.00	0.12
60	140	-0.1	0.1	0.00	0.00	0.08	0.00	0.07	1.11	0.00	0.00	0.10	0.00	0.07	1.16
70	130	-0.1	0.1	0.00	0.01	0.93	0.01	0.75	3.54	0.00	0.08	0.95	0.00	0.76	3.53
80	120	-0.1	0.1	0.05	0.95	3.33	0.25	2.72	6.34	0.01	0.96	3.39	0.25	2.73	6.33
90	110	-0.1	0.1	0.85	3.30	5.59	1.61	4.62	7.36	0.85	3.30	5.59	1.60	4.62	7.36
50	150	0.1	-0.1	0.00	0.00	0.00	0.00	0.00	0.11	0.00	0.00	0.00	0.00	0.00	0.12
60	140	0.1	-0.1	0.00	0.00	0.08	0.00	0.07	1.11	0.00	0.00	0.10	0.00	0.07	1.16
70	130	0.1	-0.1	0.00	0.01	0.93	0.01	0.75	3.54	0.00	0.08	0.95	0.00	0.76	3.53
80	120	0.1	-0.1	0.05	0.95	3.33	0.25	2.72	6.34	0.01	0.96	3.39	0.25	2.73	6.33
90	110	0.1	-0.1	0.85	3.30	5.59	1.61	4.62	7.36	0.85	3.30	5.59	1.60	4.62	7.36
			RMSE:	0.12	0.17	0.20	0.21	0.33	0.42	0.12	0.16	0.20	0.21	0.33	0.42
			RMSE:						0.2628						0.2621

				Linearin model-Daily recording (25-second average simulation time)						Naïve MC-Daily recording: 10 batches of 6,904 simulations (220-second average simulation time)						Naïve MC-Daily recording 69,038 simulations (220-second average simulation time)					
				T=0.25			T=0.50			T=0.25			T=0.50			T=0.25			T=0.50		
				σ			σ			σ			σ			σ			σ		
L	U	δ1	δ2	0.15	0.25	0.35	0.15	0.25	0.35	0.15	0.25	0.35	0.15	0.25	0.35	0.15	0.25	0.35	0.15	0.25	0.35
50	150	0	0	0.00	0.00	0.00	0.01	0.00	0.15	0.00	0.00	0.00	0.00	0.01	0.14	0.01	0.00	0.00	0.01	0.00	0.20
60	140	0	0	0.00	0.00	0.09	0.00	0.15	1.10	0.01	0.00	0.07	0.01	0.06	1.12	0.00	0.00	0.05	0.00	0.05	1.09
70	130	0	0	0.05	0.03	0.99	0.06	0.79	3.48	0.00	0.06	0.92	0.00	0.70	3.50	0.00	0.07	0.94	0.01	0.73	3.49
80	120	0	0	0.01	0.95	3.43	0.25	2.79	6.33	0.01	0.98	3.39	0.25	2.74	6.32	0.01	0.95	3.41	0.21	2.74	6.32
90	110	0	0	0.86	3.30	5.59	1.63	4.60	7.35	0.86	3.30	5.59	1.61	4.61	7.36	0.85	3.31	5.58	1.60	4.61	7.36
50	150	-0.1	0.1	0.00	0.00	0.00	0.01	0.00	0.15	0.00	0.00	0.00	0.00	0.01	0.14	0.01	0.00	0.00	0.01	0.00	0.20
60	140	-0.1	0.1	0.00	0.00	0.09	0.00	0.15	1.10	0.01	0.00	0.07	0.01	0.06	1.12	0.00	0.00	0.05	0.00	0.05	1.09
70	130	-0.1	0.1	0.05	0.03	0.99	0.06	0.79	3.48	0.00	0.06	0.92	0.00	0.70	3.50	0.00	0.07	0.94	0.01	0.73	3.49
80	120	-0.1	0.1	0.01	0.95	3.43	0.25	2.79	6.33	0.01	0.98	3.39	0.25	2.74	6.32	0.01	0.95	3.41	0.21	2.74	6.32
90	110	-0.1	0.1	0.86	3.30	5.59	1.63	4.60	7.35	0.86	3.30	5.59	1.61	4.61	7.36	0.85	3.31	5.58	1.60	4.61	7.36
50	150	0.1	-0.1	0.00	0.00	0.00	0.01	0.00	0.15	0.00	0.00	0.00	0.00	0.01	0.14	0.01	0.00	0.00	0.01	0.00	0.20
60	140	0.1	-0.1	0.00	0.00	0.09	0.00	0.15	1.10	0.01	0.00	0.07	0.01	0.06	1.12	0.00	0.00	0.05	0.00	0.05	1.09
70	130	0.1	-0.1	0.05	0.03	0.99	0.06	0.79	3.48	0.00	0.06	0.92	0.00	0.70	3.50	0.00	0.07	0.94	0.01	0.73	3.49
80	120	0.1	-0.1	0.01	0.95	3.43	0.25	2.79	6.33	0.01	0.98	3.39	0.25	2.74	6.32	0.01	0.95	3.41	0.21	2.74	6.32
90	110	0.1	-0.1	0.86	3.30	5.59	1.63	4.60	7.35	0.86	3.30	5.59	1.61	4.61	7.36	0.85	3.31	5.58	1.60	4.61	7.36
			RMSE:	0.13	0.16	0.21	0.21	0.33	0.42	0.12	0.16	0.20	0.21	0.33	0.42	0.12	0.16	0.20	0.21	0.33	0.42
			RMSE:						0.2638						0.2625						0.2629

The Root Mean Square Error (RMSE) criteria measures the distance between option values y_t computed with Ikeda and Kunitomo (1992) model adjusted for discrete time by Broadie *et al*. (1997) and the values $\widehat{y_t}$ computed with the different MC models:

$$\mathrm{RMSE} = \sqrt{\frac{\sum_{t=1}^{n}(\widehat{y_t} - y_t)^2}{n}} \qquad (9)$$

At the bottom of Tables 3 to 6, with the lowest RMSE criteria, the Login model shows its superior accuracy over the other two incremental MC models, Expin and Linearin, when pricing double barrier call and put options, Double Barrier Up-and-Out-Down-and-Out and Double Barrier Up-and-In-Down-and-In. The second best model is the Linearin model for calls and the Expin model for puts. Compare to the two naïve MC models, the Login model is slightly more accurate than or as accurate as these models. However, what makes the Login model superior is its efficiency since it takes only 25 seconds on average to obtain the option value with this model when it takes 220 seconds to obtain the option value with the two naïve MC models, meaning that the Login model is 9 times faster for a similar degree of accuracy.

Thus, our assumptions regarding the Login model suggested in the methodology section are backed by positive results: the option price computed at each experiment carries some information on the exact value of the option. Increasing the number of trials has a deterministic effect on the quality of the information obtained at each simulation as stated in the central limit theorem (Berry-Esseen theorem, 1941, 1942, 1956). We focus on the limiting behavior of S_n as N approaches infinity where S_n is the sample average of $\{X_1, ..., X_n\}$ random sample of size N - that is, a sequence of independent and identically distributed random variables drawn from distributions of expected values given by μ and finite variances given by σ^2. As a direct application of the central limit theorem, the Monte Carlo standard error of option price estimator decreases at a rate of $N^{(-1/2)}$ where N is the total number of i.i.d. trials. On top of the central limit theorem and the law of large numbers (Tchebichef, 1846), our findings suggest that the way we increase the number of trials has a deterministic effect on the quality of the information obtained at each simulation. We choose to increment the number of trials at a logarithmic rate, compared to exponential or linear rates and we show that the way the number of trials is incremented has a deterministic effect.

Furthermore, we show that the compound information obtained from averaging the results using a weighted average obtained from several experiments offers the same information (an equivalent degree of accuracy) about the exact option price than the information carried by a single experiment repeated a great number of times (in our paper 69,038 times) but the way the experiment is conducted with the Login model leads to a faster way (9 times faster) of getting the solution (the option price). Lagoze and Van de Sompel (2007) confirm this way of thinking: 'compound information objects are aggregations of distinct information units that when combined form a logical whole.'

We may explain the deterministic behavior of the logarithmic increment by the way pseudo-random numbers are generated and by their limited nature of randomness. The way we increment the number of trials may either 1) add a degree of randomness to the stochastic process of the underlying simulated asset or 2) capture the non-randomness nature of the series of pseudo-random numbers with the help of the logarithmic model which catches the interrelation between experiments through the sequence of experiments or 3) detect a relationship between the nature of the results and the number of trials leading to these results.

Whatever the inner explanation of the Login model, the added value of the model is empirically confirmed by our results. We show that *increasing* the number of trials of each experiment (e.g. 1,000, 3,709, up to 10,000 trials for a total of 69,038 trials) converges quicker in terms of execution time of the program than setting the number of experiments constant (e.g. 10 experiments of 6,904 trials) or than relying on one experiment only of 69,038 trials. In addition, we show that by increasing the number of trials at a logarithmic rate rather than at linear or exponential rates leads to better results.

5. Conclusion

The objective of variance-reduction techniques is to speed up MC simulations. We present an original variance-reduction technique that drastically reduces the execution time of the program, 9 times faster than naïve MC simulations for the same degree of accuracy. This variance-reduction technique is called Logarithmic Increment (Login) model that we apply to the pricing of European double barrier options monitored in discrete time. We test the accuracy of the Login model with an analytical solution (Ikeda and Kunitomo 1992) adjusted for discretization by Broadie et al. (1997) and a naïve numerical model using MC simulation presented by Clewlow and Strickland (2000). The Login algorithm repeats each experiment with an increasing number of trials at a logarithmic rate and calculates a weighted average of the results (i.e. options values). We assume that the option value computed at each experiment carries some information on the exact value of the option. Increasing the number of trials has a deterministic effect on the quality of the information obtained at each simulation. Incrementing the number of trials at a logarithmic rate, rather than exponential or linear rate has a synergic effect on the quality of the compound information.

The method is simple and can be applied to the pricing of plain-vanilla or more complex derivatives, easing and speeding the valuation step significantly. Broadening the application fields, the Login model can be applied in Physics for examples to the simulation of electromagnetic waves or electrons where numerical simulations of wave propagation, of conductance of electrons across double potential barrier or of diode with a resonant-tunneling structure, are involved.

6. Biography

Pierre Rostan worked for more than 12 years in industry, having developed expertise in derivatives and risk management. He has been New Products Manager in the R&D Department of the Montreal Exchange, Canada; Risk Consultant at APT Consulting Paris and Analyst at Clearstream Luxembourg. He holds a PhD in Administration from the University of Quebec, Canada. His major areas of research are numerical methods applied to derivatives and yield curve forecasting. As Associate Professor at the American University in Cairo, Egypt, he teaches courses in the area of financial markets.

Alexandra Rostan has international experience in financial analysis and consulting (Derivatives Analyst at the Montreal Exchange, Consultant in Business Creation in Paris, France) as well as experience in teaching on four continents: at the University of Quebec, Canada; SRMS International Business School, Lucknow, India; Royal Melbourne Institute of Technology, Vietnam; Montpellier Business School, France and American University, Cairo, Egypt. She holds a Master 2 in Management from the University of Nantes, France. Her major topics of research focus on derivative products.

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