A NOTE ON BOND PRICES IN THE VASICEK MODEL

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Abstract

We consider the Vasicek model for interest rates and show that it is possible to transform the governing partial differential equation into the more well-known diffusion equation and thereby use solutions of the diffusion equation to construct solutions to the Vasicek equation.

Keywords: Bond pricing, Vasicek model, Interest rate modeling.

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1 Introduction

The Vasicek model [9] was the one of the first interest rate models to incorporate a stochastic interest rate, modeling the short rate as a diffusion process with constant parameters, and as such was a special case of the Ornstein-Uhlenbeck process with constant volatility, so that the short rate was both Gaussian and Markovian. This pioneering model was mean reverting, which is a desirable property for an interest rate model and can fit historical data [10], but has the undesirable feature that interest rates may become negative. The Vasicek model remains popular amongst academic practitioners because it is highly tractable and it is possible to find closed form expressions for many interest rate securities using this model, with for example Jamshidian [3] showing how to price both coupon and zero coupon bonds under this model. In a recent paper, Mamon [6] discussed three approaches to obtaining Jamshidian's closed-form solution for a zero coupon bond under the Vasicek model. The Vasicek model has also been used to model the interest rate element of convertible securities [4, 5].

There is actually a fairly simple reason why the Vasicek model is so tractable: it is straightforward to transform the governing partial differential equation (PDE) for this model into the diffusion equation. The details of this transformation will be given in the next section, where we present our analysis.

2 Analysis

To find the price $V(r, t)$ of a security dependent on a stochastic spot interest rate $r(t)$, it is necessary to model the behavior of that interest rate, and to do so, it is usual to assume that r obeys the stochastic differential equation,

$$dr = u(r, t)dt + w(r, t)dX, \tag{1}$$

where dX is normally distributed with zero mean and variance dt and w is the volatility. By constructing a risk neutral portfolio, it can be shown that the price of the security obeys the partial differential equation

$$\frac{\partial V}{\partial t} + \frac{w^2}{2}\frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV = 0, \tag{2}$$

where $\lambda(r, t)$ is the market price of interest rate risk, and $u - \lambda w$ is the risk adjusted drift. This equation is valid for times $t \leq T$, where T is the maturity of the security. The derivation of (2) can be found in for example [10], with a more detailed discussion in [2], and this equation governs the behavior of all interest rate securities: the boundary and initial conditions rather than the PDE differentiate amongst them [8]. As noted in [10], we can interpret the solution of (2) as the expected present value of all cashflows, but this expectation is not with respect to the real random variable given by (1) but rather with respect to the risk-neutral variable. The risk-neutral spot rate obeys

$$dr = \big(u(r,t) - \lambda(r,t)\, w(r,t)\big)dt + w(r,t)dX, \tag{3}$$

with $u - \lambda w$ the risk adjusted drift, as mentioned above.

There are a number of popular interest rate models, and several of these are special cases of the general affine model, for which $u - \lambda w = a(t) - b(t)r$ and $w = (c(t)r - d(t))^{1/2}$; a table of these special cases can be found in §46.2 of [10]. For these models, the equation for the risk-neutral spot rate (3) becomes

$$dr = (a(t) - b(t)r)dt + (c(t)r - d(t))^{1/2}dX. \tag{4}$$

One popular model is the Vasicek model [9], which was one of the first interest rate models to incorporate a stochastic interest rate. For this model, $u - \lambda w = a - br$ and $w = \sigma$, with a, b and σ constants rather than functions of time, so that the risk-neutral spot rate obeys

$$dr = (a - br)dt + \sigma dX, \tag{5}$$

where a, b and σ are constants. This model is mean-reverting, with the interest rate pulled to a level a/b at a rate b, together with a normally distributed stochastic term σdX. The pricing equation (2) becomes

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial r^2} + (a - br)\frac{\partial V}{\partial r} - rV = 0. \tag{6}$$

This model is mean-reverting to a constant level $r = a/b$, which is a desirable property for interest rates, and is popular amongst academic practitioners because it is highly tractable and it is possible to find closed form expressions for many interest rate derivatives using this model.

There is actually a reason why the Vasicek is so tractable: it is possible to transform (6) into the heat conduction (or diffusion) equation. This is a property which the Vasicek PDE shares with the much more well-known Black-Scholes-Merton PDE [1,7] which governs the price of equity options. To transform (6) into the diffusion equation, we make the transformation

$$V(r,t) = \exp\left[\left(\frac{\sigma^2}{2b^2} - \frac{a}{b}\right)(T-t) - \frac{r}{b} + \frac{a}{b^2}\right] v(x,\tau), \tag{7}$$

where we have introduced the new variables

$$\tau = 1 - e^{-2b(T-t)}$$

$$x = \frac{2\sqrt{b}}{\sigma}\left[r - \frac{a}{b} + \frac{\sigma^2}{b^2}\right] e^{-b(T-t)}, \tag{8}$$

which we can invert,

$$r = \frac{a}{b} - \frac{\sigma^2}{b^2} + \frac{\sigma x}{2\sqrt{b(1-\tau)}}$$

$$t = T + \frac{\ln(1-\tau)}{2b}. \tag{9}$$

It is worth noting that the new spatial coordinate x depends on time as well as on r. Using this transformation in (6), we arrive at the diffusion equation,

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2}, \tag{10}$$

which governs one-dimensional heat conduction. If $v(x, 0)$ is known at $\tau = 0$, we can write down an expression for $v(x, \tau)$ for $\tau > 0$ using a Green's function,

$$v(x,\tau) = \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{\infty} v(z,0)\, e^{-\frac{(x-z)^2}{4\tau}}\, dz. \tag{11}$$

Using this, we can write a solution to (6) in the original variables,

$$V\left(r + \frac{a}{b} - \frac{\sigma^2}{b^2}, t\right) = \sqrt{\frac{b}{\pi\sigma^2(1 - e^{-2b(T-t)})}} \exp\left[\left(\frac{\sigma^2}{2b^2} - \frac{a}{b}\right)(T-t)\right]$$

$$\times \int_{-\infty}^{\infty} V(\tilde{r} + \frac{a}{b} - \frac{\sigma^2}{b^2}, T) \exp\left[\frac{\tilde{r} - r}{b} - \frac{b\left(\tilde{r} - re^{-b(T-t)}\right)^2}{\sigma^2(1 - e^{-2b(T-t)})}\right] d\tilde{r}. \tag{12}$$

It is straightforward to use the formula (12) to price both bonds and European interest rate derivatives under the Vasicek model. If we apply (12) to a zero coupon bond, for which the pay-off at maturity is $V(r, T) = 1$, we arrive at the well-known expression,

$$V(r,t) = \exp\left[\left(\frac{\sigma^2}{2b^2} - \frac{a}{b}\right)\left(T - t + \frac{1 - e^{-b(T-t)}}{b}\right)\right]$$

$$\times \exp\left[-\frac{\sigma^2\left(1 - e^{-b(T-t)}\right)^2}{4b^3} - \frac{r\left(1 - e^{-b(T-t)}\right)}{b}\right]. \tag{13}$$

Similarly, the price of a European interest rate call with strike X can be obtained by evaluating (12) with the pay-off at maturity of $V(r, T) = \max(r - X, 0)$ and that of put by using the pay-off $V(r, T) = \max(X - r, 0)$.

3 Discussion

In the previous section, we studied the Vasicek PDE (6) which governs the price of interest rate securities under the Vasicek model [9]. We showed that it is possible to transform (6) into the diffusion equation (10), which is a property which the Vasicek model shares with the Black-Scholes-Merton model governing the price of equity options. According to legend, the existence of this transformation for the Black-Scholes-Merton PDE was the key to arriving at the celebrated Black-Scholes-Merton option pricing formula. Similarly, for the Vasicek model, the transformation presented here (7)-(9) enables us to use the Green's function solution of the diffusion equation (11) together with the reverse transformation to arrive at a Green's function solution (12) for the Vasicek model. The existence of this transformation is the reason the Vasicek model is so tractable which in turn is why it is so popular with academic practitioners.

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